Int. J. Solids Structures, 1976, Vol. 12, pp. 107–116. Pergamon Press. Printed in Great Britain

ON BOUNDS FOR PRIMARY CREEP IN SYMMETRIC PRESSURE VESSELS

WARREN S. EDELSTEIN
Department of Mathematics, Illinois Institute of Technology,
Chicago, IL 60616, U.S.A.

(*Received* 6 *March* 1975; *revised* 2 *July* 1975)

Abstract—The field equations governing primary creep in spherical and incompressible cylindrical pressure vessels subject to a non-decreasing internal pressure are reduced to a single equation in the effective stress. The effects of elastic strains are included. Using this equation, bounds and monotonicity properties are established for the effective stress at any point in the body at any time. It is also shown that these results imply upper and lower bounds for the displacement. These bounds are stated explicitly for the displacement of the outer surface of the vessel.

1. INTRODUCTION

This work, although self-contained, is intended as a sequel to [1]. That paper dealt with the problem of quasistatic creep in spherical and cylindrical pressure vessels subject to a nondecreasing internal pressure. Small total strains were assumed, and the effect of elastic strains was included as in (2.1) below. A creep flow law was assumed of the form (2.3b) below with $m = 0$. This is a widely used three-dimensional generalization of the Norton power law. With these assumptions, a set of inequalities was derived in Section 3 of [1] giving bounds and monotonicity properties for various quantities of physical interest.

In the present paper, we consider precisely the same boundary value problems described above but for the more general case $m \geq 0$. When $m > 0$, we have so-called *primary* or *strain-hardening* creep as opposed to the case of *secondary* creep in which $m = 0$. Our purpose is to extend to the theory of primary creep the inequalities of Section 3 of [1].

In Section 2 below, the system of nonlinear field equations governing an infinite incompressible hollow cylinder is reduced to a single nonlinear integral eqn (2.28) in the effective stress σ. This equation appears to be new. The analogous eqn (2.30) for the hollow sphere problem is then stated without proof. Both (2.28) and (2.30) are then combined in a unified eqn (2.34) on which further investigations are based. A unified displacement eqn (2.35) governing the radial displacement of the outer surface of the body is also given.

In Section 3, we obtain our main results. These are the inequalities (3.10), which give upper and lower bounds for the effective stress, (3.11) and (3.12), which furnish bounds for the displacement history of the outer surface, and (3.13) in which a bound is given for the magnitude of the gradient of the effective stress. It will also become apparent to the reader that the analysis in this paper readily implies bounds for many other quantities of physical interest, e.g. the displacement history of the inner surface of the vessel. Also the proof of the above results involves the discovery of monotonicity results, (3.4) and (3.5), which are of interest in their own right.

Our method of derivation is conceptually the same as that used in [1], in that it is also based on the use of simple differential inequalities. However the proofs in the present paper are somewhat more complicated than their counterparts in [1] due to the extra degree of nonlinearity introduced by allowing $m > 0$.

Finally, we remark that for the special case in which the spherical or incompressible cylindrical pressure vessel has been subject to an internal pressure which is constant in time, Einarsson[2] states upper and lower bounds for a quantity y which imply bounds for the displacements and stresses. However in the case $m > 0$, no proof for these bounds is given. For $m = 0$, a proof is given in a subsequent paper[3].

In Section 4, upper and lower bounds for the displacement of the outer surface of a cylinder are evaluated for different pressures and times in the case of 12% Cr steel at 850°F. Also various technological and theoretical applications are suggested for the results of Section 3. This section concludes with a new formal derivation of the well-known formula (4.4) for the limiting state $\sigma(r, \infty)$ using the unified eqn (2.34) under the assumption that $P(t)$ has a finite limit $P(\infty)$.

2. DERIVATION OF EQUATIONS

As in [1], we assume that the infinitesimal strains ϵ_{ij} have the form

$$\epsilon_{ij} = \epsilon_{ij}^{(e)} + \epsilon_{ij}^{(c)} \tag{2.1}$$

where $\epsilon_{ij}^{(e)}$ and $\epsilon_{ij}^{(c)}$ denote, respectively, elastic strains and creep strains. These are related to the stresses σ_{ij} by the equations†

$$\epsilon_{ij}^{(e)} = \frac{1}{E}[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}], \tag{2.2}$$

$$\epsilon_{ij}^{(c)}\big|_{t=0} = 0 \tag{2.3a}$$

$$\frac{\partial \epsilon_{ij}^{(c)}}{\partial t} = \frac{3}{2}\frac{K\sigma_e^{\,n-1}}{[\epsilon_e^{(c)}]^m}\cdot s_{ij}, \quad t>0. \tag{2.3b}$$

Here, s_{ij} stands for the deviatoric components of the stress, σ_e is the effective stress and $\epsilon_e^{(c)}$ is the effective creep strain. They are defined by the formulas

$$s_{ij} = \sigma_{ij} - \frac{\delta_{ij}}{3}\sigma_{kk}, \tag{2.4}$$

$$\sigma_e = \sqrt{\frac{3}{2}s_{ij}s_{ij}}, \quad \epsilon_e^{(c)} = \sqrt{\frac{2}{3}\epsilon_{ij}^{(c)}\epsilon_{ij}^{(c)}}. \tag{2.5}$$

Notice that (2.3) and (2.4) imply that the creep deformation is isochoric.

E and ν are Young's modulus and Poisson's ratio respectively. They are assumed to satisfy the inequalities

$$E>0, \quad -1<\nu\le\frac{1}{2}.$$

The creep constants K, m, n appearing in (2.3) are subject to the restrictions‡

$$K\ge 0, \quad m\ge 0, \quad n\ge m+1. \tag{2.6}$$

The constitutive law defined by the above equations is widely used in creep theory and is given in [4].

We shall now sketch the derivation of the equation which governs the effective stress in an infinite hollow incompressible right circular cylinder of inner radius a and outer radius b subject to a nondecreasing internal pressure $p(t)>0$ with $\dot{p}(t)\ge 0$ and zero body force. It is assumed that the response is cylindrically symmetric and in a state of plane strain. In cylindrical coordinates r, θ, z where the z axis is taken to be the axis of the cylinder and the components of the displacement vector are denoted u_r, u_θ, u_z, these assumptions take the form

$$u_r = u(r,t), \quad u_\theta = u_z = 0. \tag{2.7}$$

It follows that

$$\epsilon_r = \frac{\partial u}{\partial r}, \quad \epsilon_\theta = \frac{u}{r}, \quad \epsilon_z = \epsilon_{r\theta} = \epsilon_{rz} = \epsilon_{\theta z} = 0. \tag{2.8}$$

The first two equations of (2.8) imply the compatibility equation

$$\epsilon_r = \frac{\partial}{\partial r}[r\epsilon_\theta]. \tag{2.9}$$

†Subscripts have the range 1, 2, 3, δ_{ij} stands for Kronecker delta, and summation over repeated indices is implied. We shall also use a superposed dot to denote differentiation with respect to time.

‡The restriction $n\ge m+1$ also appears in [2].

Also, the assumption of incompressibility takes the form

$$\epsilon_r + \epsilon_\theta = 0. \tag{2.10}$$

It follows from (2.9) and (2.10) that

$$\epsilon_r = \frac{-\sqrt{3}}{2}\frac{f(t)}{r^2} \tag{2.11}$$

for some function $f(t)$.

Incompressibility also means that $\nu = \frac{1}{2}$. This fact, together with (2.1), (2.2), (2.3) leads to the strain–stress relations

$$\dot{\epsilon}_{ij} = \frac{3}{2E}\dot{s}_{ij} + \frac{3K}{2}\frac{\sigma_e^{n-1}}{[\epsilon_e^{(c)}]^m} s_{ij} \quad (t > 0), \tag{2.12}$$

$$\epsilon_{ij}\big|_{t=0} = \frac{3}{2E} s_{ij}\big|_{t=0}. \tag{2.13}$$

If it is assumed that the coefficient $\sigma_e^{n-1}/[\epsilon_e^{(c)}]^m$ in (2.12) is sufficiently well-behaved, then (2.8), (2.12) and (2.13) imply that

$$s_z = s_{r\theta} = s_{rz} = s_{\theta z} = 0. \tag{2.14}$$

It then follows from (2.3) and (2.14) that

$$\epsilon_z^{(c)} = \epsilon_{r\theta}^{(c)} = \epsilon_{rz}^{(c)} = \epsilon_{\theta z}^{(c)} = 0. \tag{2.15}$$

Due to (2.15) and the fact that the creep deformation is isochoric, the effective creep strain takes the form

$$\epsilon_e^{(c)} = \frac{2}{\sqrt{3}}|\epsilon_r^{(c)}| = \frac{2}{\sqrt{3}}|\epsilon_\theta^{(c)}|. \tag{2.16}$$

On physical grounds, we restrict ourselves to solutions of the boundary value problem for which

$$\epsilon_r^{(c)} < 0, \quad \epsilon_\theta^{(c)} > 0 \quad (t > 0). \tag{2.17}$$

Equation (2.16) then becomes

$$\epsilon_e^{(c)} = -\frac{2}{\sqrt{3}}\epsilon_r^{(c)} = \frac{2}{\sqrt{3}}\epsilon_\theta^{(c)}. \tag{2.18}$$

Let us now consider the strain–stress relation (2.3) for the case $\epsilon_r^{(c)}$. Using (2.18), we may integrate this initial value problem to get

$$-\frac{2}{\sqrt{3}}\epsilon_r^{(c)} = \left[-\sqrt{3}(m+1)K\int_0^t \sigma_e^{n-1}s_r \, \mathrm{d}\tau\right]^{1/(m+1)}, \tag{2.19}$$

where the positive real root is understood. In order to see that the existence of such a root is physically plausible, we notice that by (2.4) and (2.14),

$$2s_\theta = -2s_r = \sigma_\theta - \sigma_r, \tag{2.20}$$

and it is reasonable for this type of loading that

$$\sigma_\theta > 0, \quad \sigma_r < 0 \quad (t > 0).$$

In this case, (2.5), (2.14) and (2.20) imply that

$$\sigma_e = \frac{\sqrt{3}}{2}(\sigma_\theta - \sigma_r) = -\sqrt{3}\, s_r \equiv \sigma. \tag{2.21}$$

Equations (2.21), (2.19), (2.11), (2.1) and (2.2) (with $\nu = \frac{1}{2}$) imply that the *effective stress*, now denoted σ, satisfies the equation

$$\frac{f(t)}{r^2} = \frac{\sigma}{E} + \left[(m+1)K \int_0^t \sigma^n \, \mathrm{d}\tau\right]^{1/(m+1)}. \tag{2.22}$$

The function $f(t)$ can be eliminated by means of the equilibrium equation

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{2.23}$$

and the boundary conditions

$$\sigma_r(a, t) = -p(t), \quad \sigma_r(b, t) = 0. \tag{2.24}$$

In fact, from (2.21), (2.23) and (2.24) it follows that

$$\frac{2}{\sqrt{3}} \int_a^b \sigma(\xi, t) \frac{\mathrm{d}\xi}{\xi} = p(t). \tag{2.25}$$

Therefore, if we multiply (2.22) by r^{-1} and integrate with respect to r from a to b, we obtain the following representation for f in terms of σ and p:

$$f(t) = \frac{\beta_c}{E}\left(\frac{\sqrt{3}p(t)}{2} + \mu_c \int_a^b \left[\int_0^t \sigma^n(\xi, \tau)\, \mathrm{d}\tau\right]^{1/(m+1)} \frac{\mathrm{d}\xi}{\xi}\right) \tag{2.26}$$

where

$$\beta_c^{-1} = \int_a^b \frac{\mathrm{d}r}{r^3}, \quad \mu_c = E[(m+1)K]^{1/(m+1)}. \tag{2.27}$$

Plugging (2.26) into (2.22), we find the effective stress equation for the incompressible cylinder,

$$\sigma(r, t) = \frac{\sqrt{3}\beta_c p(t)}{2r^2} + \mu_c\left(\frac{\beta_c}{r^2}\int_a^b \left[\int_0^t \sigma^n(\xi, \tau)\, \mathrm{d}\tau\right]^{1/(m+1)} \frac{\mathrm{d}\xi}{\xi} - \left[\int_0^t \sigma^n(r, \tau)\, \mathrm{d}\tau\right]^{1/(m+1)}\right). \tag{2.28}$$

Notice that in the case of secondary creep, i.e. $m = 0$, (2.28) reduces to eqn (2.32) of [1].

Using (2.8), (2.10), (2.11) and (2.26) we obtain the following representation for u in terms of σ:

$$u(r, t) = \frac{\sqrt{3}\beta_c}{2Er}\left(\frac{\sqrt{3}p(t)}{2} + \mu_c \int_a^b \left[\int_0^t \sigma^n(\xi, \tau)\, \mathrm{d}\tau\right]^{1/(m+1)} \frac{\mathrm{d}\xi}{\xi}\right). \tag{2.29}$$

Let us now consider the case of a hollow sphere with inner radius a and outer radius b subject to zero body force and a non-decreasing internal pressure $p > 0$. We again denote by σ and u the effective stress and radial displacement respectively. In this case, the equations analogous to (2.28) and (2.29) are

$$\sigma(r, t) = \frac{\beta_s p(t)}{2r^3} + \mu_s\left(\frac{\beta_s}{r^3}\int_a^b \left[\int_0^t \sigma^n(\xi, \tau)\, \mathrm{d}\tau\right]^{1/(m+1)} \frac{\mathrm{d}\xi}{\xi} - \left[\int_0^t \sigma^n(r, \tau)\, \mathrm{d}\tau\right]^{1/(m+1)}\right), \tag{2.30}$$

$$u(r,t)=\frac{r}{E}\left[(1-\nu)\sigma(r,t)-(1-2\nu)p(t)+2(1-2\nu)\int_a^r\sigma(\xi,t)\frac{\mathrm{d}\xi}{\xi}\right]$$
$$+\frac{r}{2}[K(m+1)]^{1/(m+1)}\left[\int_0^t\sigma^n(r,\tau)\,\mathrm{d}\tau\right]^{1/(m+1)}, \tag{2.31}$$

where

$$\beta_s^{-1}=\int_a^b\frac{\mathrm{d}r}{r^4},\quad \mu_s=\frac{E}{2(1-\nu)}[K(m+1)]^{1/(m+1)}. \tag{2.32}$$

Equations (2.30) and (2.31) correspond respectively to (2.28) and (2.29) of [1]. Notice that if (2.30) is multiplied by r^{-1} and integrated with respect to r from a to b we obtain the result that

$$\int_a^b\sigma(\xi,t)\frac{\mathrm{d}\xi}{\xi}=\frac{p(t)}{2}.$$

This is the spherical analogue to eqn (2.25) above. Using it together with (2.30) we obtain the following representation for the displacement history of the outer surface of the spherical shell:

$$u(b,t)=\frac{(1-\nu)\beta_s}{Eb^2}\left\{\frac{p(t)}{2}+\mu_s\int_a^b\left[\int_0^t\sigma^n(\xi,\tau)\,\mathrm{d}\tau\right]^{1/(m+1)}\frac{\mathrm{d}\xi}{\xi}\right\}. \tag{2.33}$$

Consider the equations

$$\sigma(r,t)=\frac{\beta P(t)}{r^j}+\mu\left(\frac{\beta}{r^j}\int_a^b\left[\int_0^t\sigma^n(\xi,\tau)\,\mathrm{d}\tau\right]^{1/(m+1)}\frac{\mathrm{d}\xi}{\xi}-\left[\int_0^t\sigma^n(r,\tau)\,\mathrm{d}\tau\right]^{1/(m+1)}\right), \tag{2.34}$$

for $a\le r\le b$, $t\ge 0$, and

$$u(b,t)=\frac{\kappa\beta}{b^{j-1}}\left(P(t)+\mu\int_a^b\left[\int_0^t\sigma^n(\xi,\tau)\,\mathrm{d}\tau\right]^{1/(m+1)}\frac{\mathrm{d}\xi}{\xi}\right), \tag{2.35}$$

for $t\ge 0$ where

$$\beta^{-1}=\int_a^b\frac{\mathrm{d}\xi}{\xi^{j+1}}. \tag{2.36}$$

If we take

$$j=2,\quad P=\frac{\sqrt{3}}{2}p,\quad \mu=\mu_c,\quad \kappa=\frac{\sqrt{3}}{2E} \tag{2.37}$$

then (2.34), (2.35) reduce to the *cylinder* eqns (2.28) and (2.29), the latter being evaluated at $r=b$. For

$$j=3,\quad P=\frac{1}{2}p,\quad \mu=\mu_s,\quad \kappa=\frac{1-\nu}{E} \tag{2.38}$$

(2.34), (2.35) become the sphere eqns (2.30), (2.33). It is consistent with these two cases to assume that for $t\ge 0$, $P(t)$ is continuously differentiable and

$$P>0,\quad \dot{P}\ge 0,\quad \mu\ge 0,\quad \kappa>0. \tag{2.39}$$

3. BOUNDS AND MONOTONICITY PROPERTIES

We consider only solutions $\sigma(r,t)$ of (2.34) which are strictly positive and twice continuously differentiable on $[a,b]\times[0,\infty)$. The assumption of strict positiveness is consistent with the derivation of (2.34) and with the physics of the problem.

Let

$$\phi(r,t)=\int_0^t \sigma^n(r,\tau)\,\mathrm{d}\tau. \tag{3.1}$$

With this notation, (2.34) and (2.35) become

$$\sigma(r,t)=\frac{\beta P(t)}{r^j}+\mu\left(\frac{\beta}{r^j}\int_a^b \phi^{1/(m+1)}(\xi,t)\frac{\mathrm{d}\xi}{\xi}-\phi^{1/(m+1)}(r,t)\right), \tag{3.2}$$

$$u(b,t)=\frac{\kappa\beta}{b^{j-1}}\left(P(t)+\mu\int_a^b \phi^{1/(m+1)}(\xi,t)\frac{\mathrm{d}\xi}{\xi}\right). \tag{3.3}$$

The main results of this paper are implied by the following inequalities:

$$\frac{\partial\sigma}{\partial r}<0 \quad (t>0) \tag{3.4}$$

and

$$\text{(a)}\ \frac{\partial}{\partial r}(r^j\sigma)\geq 0,\quad \text{(b)}\ \frac{\partial}{\partial r}(r^j\phi^{1/(m+1)})\leq 0 \quad (t>0). \tag{3.5}$$

In order to see this, let us first suppose that (3.4) and (3.5) have already been established. Then (3.5) (b) applied to (3.2) and (3.3) yields the inequalities

$$\sigma(r,t)\leq\frac{\beta P(t)}{r^j}+\frac{\mu}{r^j}[a^j\phi^{1/(m+1)}(a,t)-r^j\phi^{1/(m+1)}(r,t)], \tag{3.6}$$

$$\sigma(r,t)\geq\frac{\beta P(t)}{r^j}+\frac{\mu}{r^j}[b^j\phi^{1/(m+1)}(b,t)-r^j\phi^{1/(m+1)}(r,t)], \tag{3.7}$$

$$u(b,t)\leq\frac{\kappa\beta}{b^{j-1}}[P(t)+\mu\beta^{-1}a^j\phi^{1/(m+1)}(a,t)], \tag{3.8}$$

$$u(b,t)\geq\frac{\kappa\beta}{b^{j-1}}[P(t)+\mu\beta^{-1}b^j\phi^{1/(m+1)}(b,t)]. \tag{3.9}$$

Inequalities (3.4), (3.6) and (3.7) immediately yield the following bounds for the effective stress:

$$\frac{\beta P(t)}{b^j}\leq\sigma(r,t)\leq\frac{\beta P(t)}{a^j}(a\leq r\leq b,\ t\geq 0). \tag{3.10}$$

These inequalities, together with (3.8) and (3.9) imply the displacement estimates

$$u(b,t)\leq\frac{\kappa\beta}{b^{j-1}}\left\{P(t)+\mu\left(\frac{\beta}{a^j}\right)^{(n-m-1)/(m+1)}\left[\int_0^t P^n(\tau)\,\mathrm{d}\tau\right]^{1/(m+1)}\right\}, \tag{3.11}$$

$$u(b,t)\geq\frac{\kappa\beta}{b^{j-1}}\left\{P(t)+\mu\left(\frac{\beta}{b^j}\right)^{(n-m-1)/(m+1)}\left[\int_0^t P^n(\tau)\,\mathrm{d}\tau\right]^{1/(m+1)}\right\}. \tag{3.12}$$

Notice that these bounds converge to the exact solution as t tends to zero. In the case of secondary creep, i.e. $m=0$, they reduce to (3.9) and (3.10) of [1]. Finally, we remark that (3.4), (3.5) (a) and (3.10) furnish the following bound for the gradient of the effective stress:

$$\left|\frac{\partial\sigma}{\partial r}(r,t)\right|\leq\frac{j\beta}{a^j r}P(t)(a\leq r\leq b,\quad t>0). \tag{3.13}$$

The rest of this section is devoted to establishing (3.4) and (3.5). It turns out that both of these results depend on the inequality

$$\frac{\partial \phi}{\partial r} < 0 \quad (t > 0), \tag{3.14}$$

which will therefore be proved first. In this and subsequent analysis we shall repeatedly use the elementary fact that if $y(t)$ satisfies the differential equation

$$\dot{y} + Qy = F \quad (t > 0) \tag{3.15}$$

where $y(0+) \geq 0$ and $F \geq 0$ on $[0, \infty)$ (resp $y(0+) \leq 0$, $F \leq 0$ on $[0, \infty)$) then $y \geq 0$ on $(0, \infty)$ (resp $y \leq 0$ on $[0, \infty)$). It is assumed here that all of the quantities involved are suitably smooth functions of t. Q and F may also be allowed to have mild singularities at 0. Also, if $y(0+) \leq 0$ and $F < 0$ for $t > 0$, it follows that $y < 0$ for $t > 0$.

Let

$$\Phi(t) = P(t) + \mu \int_a^b [\phi(\xi, t)]^{1/(m+1)} \frac{d\xi}{\xi}. \tag{3.16}$$

Then

$$\Phi > 0, \quad \dot{\Phi} \geq 0 \quad (t > 0), \tag{3.17}$$

and (3.2) takes the form

$$\sigma(r, t) = \frac{\beta}{r^j} \Phi(t) - \mu \phi^{1/(m+1)}(r, t). \tag{3.18}$$

If we now differentiate (3.18) with respect to r and multiply both sides of the resulting equation by $n\sigma^{n-1}$, then, in view of the definition (3.1) for ϕ, we obtain the following initial value problem for $\partial \phi / \partial r$:

$$\frac{\partial^2 \dot{\phi}}{\partial t \partial r} + \frac{\mu n}{m+1} \sigma^{n-1} \phi^{-[m/(m+1)]} \frac{\partial \phi}{\partial r} = \frac{-jn\beta}{r^{j+1}} \sigma^{n-1} \Phi, \quad \frac{\partial \phi}{\partial r}(r, 0) = 0. \tag{3.19}$$

In view of the remarks made concerning (3.15), inequality (3.14) now follows from (3.17) and (3.19).

In order to apply this result to the proof of (3.4), we differentiate (3.18) with respect to r and t. This yields the equation

$$\frac{\partial^2 \dot{\sigma}}{\partial t \partial r} + \frac{\mu n}{m+1} \phi^{-[m/(m+1)]} \sigma^{n-1} \frac{\partial \sigma}{\partial r} = \frac{-j\beta}{r^{j+1}} \dot{\Phi} + \frac{\mu m \sigma^n}{(m+1)^2 \phi^{(2m+1)/(m+1)}} \frac{\partial \phi}{\partial r} \quad (t > 0). \tag{3.20}$$

If we adjoin the initial condition

$$\frac{\partial \sigma}{\partial r}(r, 0+) = \frac{-j\beta P(0)}{r^{j+1}}, \tag{3.21}$$

then, by virtue of (3.17), (3.14) and (2.39), the initial value problem (3.20), (3.21) implies (3.4).

In order to establish (3.5), we first notice that, by (3.18),

$$\frac{\partial}{\partial r}[r^j \sigma(r, t)] = -\mu \frac{\partial}{\partial r}[r^j \phi^{1/(m+1)}(r, t)]. \tag{3.22}$$

This equation shows that (3.5) (b) is immediate, once we demonstrate (3.5) (a). To this end, we introduce the quantity

$$\psi = \int_0^t \sigma^{n-1} \frac{\partial}{\partial r}(r^j \sigma)\, d\tau, \tag{3.23}$$

and record the fact that

$$r^j \frac{\partial \phi}{\partial r} = n\psi - njr^{j-1}\phi.$$

Then one differentiation of (3.22) with respect to time and some rearrangement of terms in the resulting equation yield the identity

$$\frac{\partial}{\partial r}(r^j \dot{\sigma}) + \frac{\mu}{m+1}\phi^{-[m/(m+1)]}\sigma^{n-1}\frac{\partial}{\partial r}(r^j\sigma) = \frac{\mu\sigma^{n-1}}{(m+1)\phi^{(2m+1)/(m+1)}}\left[\frac{mn}{m+1}\cdot\sigma\psi - n\phi\left(\frac{m}{m+1}\sigma j r^{j-1} + \frac{(n-1)}{n}r^j\frac{\partial\sigma}{\partial r}\right)\right]. \tag{3.24}$$

By (3.24), (2.6) and (3.4),

$$\frac{\partial}{\partial r}(r^j\dot{\sigma}) + \frac{\mu}{m+1}\phi^{-m/(m+1)}\sigma^{n-1}\frac{\partial}{\partial r}(r^j\sigma) \geq \frac{\mu\sigma^{n-1}}{(m+1)\phi^{(2m+1)/(m+1)}}\left[\frac{mn}{m+1}\cdot\sigma\psi - \frac{nm}{m+1}\phi\frac{\partial}{\partial r}(r^j\sigma)\right], \tag{3.25}$$

or, equivalently,

$$\frac{\partial}{\partial r}(r^j\dot{\sigma}) + \frac{(mn+m+1)}{(m+1)^2}\mu\phi^{-m/(m+1)}\sigma^{n-1}\frac{\partial}{\partial r}(r^j\sigma) \geq \frac{\mu mn\sigma^n\psi}{(m+1)^2\phi^{(2m+1)/(m+1)}}. \tag{3.26}$$

Since, by (3.22) and (3.1)

$$\frac{\partial}{\partial r}[r^j\sigma(r, 0+)] = 0,$$

the proof of (3.5) (a) will be complete once we show that

$$\psi \geq 0 \quad (t \geq 0). \tag{3.27}$$

In order to see this, we carry out the differentiation indicated on the right-hand side of (3.22) to obtain

$$\frac{\partial}{\partial r}(r^j\sigma) = -\mu\phi^{-m/(m+1)}\left[jr^{j-1}\phi + \frac{r^j}{m+1}\frac{\partial\phi}{\partial r}\right]. \tag{3.28}$$

We then use the identity

$$jr^{j-1}\phi = \psi - \frac{r^j}{n}\frac{\partial\phi}{\partial r}$$

to put (3.28) in the form

$$\frac{\partial}{\partial r}(r^j\sigma) + \mu\phi^{-m/(m+1)}\psi = \mu\phi^{-m/(m+1)}\, r^j\frac{\partial\phi}{\partial r}\left(\frac{1}{n} - \frac{1}{m+1}\right). \tag{3.29}$$

If both sides of this equation are multiplied by σ^{n-1}, we get

$$\dot{\psi} + \mu\phi^{-m/(m+1)}\sigma^{n-1}\psi = \mu\phi^{-m/(m+1)}\,\sigma^{n-1}r^j\frac{\partial\phi}{\partial r}\left(\frac{1}{n} - \frac{1}{m+1}\right).$$

This expression, together with (2.6), (3.14) and (3.23) implies (3.27).

4. EXAMPLES AND APPLICATIONS

Using numbers furnished by Hult in [5], we can apply the displacement bounds (3.11), (3.12) to the case of a hollow cylinder made out of 12% Cr steel at 850°F (454°C). With the units kg/mm^2 and hr, Hult asserts that $K = 0{\cdot}5\times10^{-20}$, $n = 7{\cdot}5$, $m = 1{\cdot}8$, $E = 16{,}200$ kg/mm^2 and $\nu = 0{\cdot}3$. For definiteness, we have chosen $a = 100$ mm, $b = 130$ mm. The results are summarized in the following two tables in which are evaluated the upper and lower bounds respectively on the displacement $u(b, t)$ of the outer surface of the cylinder for various times and constant pressure histories.

Upper Bounds (mm)

	100 hrs	200 hrs	500 hrs	1000 hrs	2000 hrs
p=5 kg/mm^2	.19	.22	.27	.32	.38
10	.82	1.00	1.31	1.63	2.04
15	2.16	2.69	3.64	4.58	5.80

Lower Bounds

	100 hrs	200 hrs	500 hrs	1000 hrs	2000 hrs
p=5 kg/mm^2	.13	.14	.16	.18	.21
10	.44	.51	.65	.78	.95
15	1.05	1.27	1.66.	2.05	2.56

Let $U(t)$, $L(t)$ denote respectively the upper and lower bounds on $u(b, t)$. It is of interest to note that with the above choice of parameters,

$$\lim_{t\to\infty}\frac{U(t)}{L(t)} = 2{\cdot}41.$$

It is hoped that the bounds (3.10), (3.11), (3.12) will find direct industrial applications. For instance, in preliminary design work on such things as pressure vessels precise answers are often not required and simple bounds on the deformation are sufficient. Also situations arise in which there are long "lead times" on the ordering of materials, and orders involving dimensions and quantities have to be placed before a detailed numerical analysis of the structure can be implemented. In this case bounds can be helpful.

On the theoretical side, they might be employed for such purposes as the derivation of more refined bounds, the establishment of existence and uniqueness theorems, and the mathematical analysis of the asymptotic behavior of solutions to problems involving primary creep.

As another application of the theory developed in Sections 2 and 3, we present a *formal* derivation of the limiting stationary state $\sigma(r, \infty)$, which is assumed to exist given that $P(t)$ has a finite limit $P(\infty)$ as $t\to\infty$. Let t_1 be such that

$$\sigma(r, t) = \sigma(r, \infty) \quad (t \geq t_1)$$

to as many significant figures as is desired. Then for $t > t_1$, (2.34) can be put in the form

$$\begin{aligned}\frac{\sigma(r,t)}{(t-t_1)^{1/(m+1)}} &= \frac{\beta P(t)}{r^j (t-t_1)^{1/(m+1)}}\\ &+ \mu\left(\frac{\beta}{r^j}\int_a^b \left[\frac{\int_0^{t_1}\sigma^n(\xi,\tau)\,d\tau}{(t-t_1)\sigma^n(\xi,\infty)}+1\right]^{1/(m+1)} \sigma^{n/(m+1)}(\xi,\infty)\frac{d\xi}{\xi}\right. \qquad (4.1)\\ &\left. - \left[\frac{\int_0^{t_1}\sigma^n(r,\tau)\,d\tau}{(t-t_1)\sigma^n(r,\infty)}+1\right]^{1/(m+1)} \sigma^{n/(m+1)}(r,\infty)\right).\end{aligned}$$

Due to (2.39) and the assumed existence of finite $P(\infty)$, P is bounded. Also, the results of Section 3 imply σ is bounded above and below for all time. Therefore, if we take the limit of (4.1) as $t \to \infty$, we obtain

$$\frac{\beta}{r^j} \int_a^b \sigma^{n/(m+1)}(\xi, \infty) \frac{d\xi}{\xi} = \sigma^{n/(m+1)}(r, \infty).$$

This shows that $\sigma(r, \infty)$ has the form

$$\sigma(r, \infty) = A r^{-[j(m+1)/n]}. \tag{4.2}$$

In order to evaluate A, we notice that (2.34) implies the identity (cf. (2.25))

$$\int_a^b \sigma(r, \infty) \frac{dr}{r} = P(\infty). \tag{4.3}$$

Substitution of the expression (4.2) for $\sigma(r, \infty)$ into (4.3) gives the result

$$\sigma(r, \infty) = \frac{j(m+1)}{n} P(\infty) [a^{-[j(m+1)/n]} - b^{-[j(m+1)/n]}]^{-1} r^{-[j(m+1)/n]}. \tag{4.4}$$

Notice that for the spherical case, this agrees with Hult's results (33) and (34) in [5], the latter having been derived under the assumption of constant pressure and negligible elastic strains.

Acknowledgements—This work was done in part while the author was a faculty research participant at the Argonne National Laboratory. The advice and encouragement of Dr. Richard Valentin and Dr. Carl Youngdahl of that laboratory are gratefully acknowledged.

REFERENCES

1. W. S. Edelstein and R. A. Valentin, *On Bounds and Limit Theorems for Secondary Creep in Symmetric Pressure Vessels. Int. J. Non-Linear Mech.*, To be published.
2. B. Einarsson, Primary creep in thickwalled shells. *J. Engng Math.* **2**, 123 (1968).
3. B. Einarsson, Numerical treatment of integro-differential equations with a certain maximum property. *Numer. Math.* **18**, 267 (1971).
4. F. K. G. Odqvist and J. Hult, *Kriechfestigkeit Metallischer Werkstoffe.* Springer, Berlin (1962).
5. J. Hult, *Primary Creep in Thickwalled Spherical Shells.* Trans. Chalmers U. of Technol. Nr 264 (1963).